C4 January 2008 CORE 4

Q1

(a)

| $x$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ |
|---|---|---|---|---|---|
| $y$ | $0$ | 1.84432 | 4.810477 | 8.87207 | $0$ |

(2)

(b) $\int_a^b y\,dx \approx \frac{1}{2}h\left[(y_0+y_n)+2(y_1+\dots+y_{n-1})\right]$

$h=\frac{\pi}{4} \Rightarrow \quad I=\frac{1}{2}\times\frac{\pi}{4}\left(0+0+2(1.84432+4.810477+8.87207)\right)$

$= 12.1948$ (4dp)

$\therefore$ Area $= 12.1948$ (4)

Q2 (a)

$(8-3x)^{\frac{1}{3}} \qquad |x|<\frac{8}{3}$

$=8^{\frac{1}{3}}\left(1-\frac{3}{8}x\right)^{\frac{1}{3}}$

$=2\left(1-\frac{3}{8}x\right)^{\frac{1}{3}}$

$=2\left(1+\frac{1}{3}\left(-\frac{3}{8}x\right)+\frac{\frac{1}{3}\times\left(\frac{1}{3}-1\right)}{2!}\left(-\frac{3}{8}x\right)^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(-\frac{3}{8}x\right)^3 \dots\right)$

$=2-\frac{1}{4}x-\frac{1}{32}x^2-\frac{5}{768}x^3$

(b) $7.7=8-3\times0.1$ so with $x=0.1$ we get:

$7.7^{\frac{1}{3}}=(8-3\times0.1)^{\frac{1}{3}}\approx 2-\frac{1}{4}(0.1)-\frac{1}{32}(0.1)^2-\frac{5}{768}(0.1)^3$

$=2-\frac{1}{40}-\frac{1}{3200}-\frac{5}{768000}$

$=1.97468099$

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Q3

Volume $=\pi\int_a^b y^2\,dx=\pi\int_a^b\frac{1}{(2x+1)^2}\,dx$

$=\pi\int_a^b(2x+1)^{-2}\,dx$

Let $u=2x+1$ limits $u(a)=2a+1$

$\frac{du}{dx}=2 \Rightarrow dx=\frac{du}{2}$ $\qquad u(b)=2b+1$

Volume $=\pi\int_{2a+1}^{2b+1}u^{-2}\frac{du}{2}$

$=\frac{\pi}{2}\int_{2a+1}^{2b+1}u^{-2}\,du$

$=\frac{\pi}{2}\left[-\frac{1}{u}\right]_{2a+1}^{2b+1}$

$=\frac{\pi}{2}\left\{\left(-\frac{1}{2b+1}\right)-\left(-\frac{1}{2a+1}\right)\right\}$

$=\frac{\pi}{2}\left(-\frac{1}{2b+1}+\frac{1}{2a+1}\right)$

$=\frac{\pi}{2}\left(\frac{2b+1-(2a+1)}{(2a+1)(2b+1)}\right)$

$=\frac{\pi}{2}\,\frac{2(b-a)}{(2a+1)(2b+1)}$

$=\frac{\pi(b-a)}{(2a+1)(2b+1)}$ (5)

Q4 (i) $I=\int\ln\left(\frac{x}{2}\right)dx$

Integrating by parts: Let $u=\ln\frac{x}{2}$ $\qquad \frac{dv}{dx}=1$

$\frac{du}{dx}=\frac{1}{2}\times\frac{1}{\frac{x}{2}}=\frac{1}{x}$ $\qquad v=x$

$I=x\ln\left(\frac{x}{2}\right)-\int\frac{1}{x}\times x\,dx$

$=x\ln\frac{x}{2}-\int dx$

$=x\ln\frac{x}{2}-x+C$

(ii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^2x\,dx$

Using $\cos 2x\equiv\cos^2x-\sin^2x$

$=1-2\sin^2x$

So, $\sin^2x=\frac{1}{2}(1-\cos 2x)$

$=\frac{1}{2}-\frac{1}{2}\cos 2x$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx$

$=\left[\frac{1}{2}x-\frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$\int\frac{1}{2}\cos 2x=\frac{1}{2}\times\frac{1}{2}\sin 2$

$=\frac{1}{4}\sin 2x$

$=\left(\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{4}\sin\left(2\times\frac{\pi}{2}\right)\right)-\left(\frac{1}{2}\left(\frac{\pi}{4}\right)-\frac{1}{4}\sin\left(2\times\frac{\pi}{4}\right)\right)$

$=\left(\frac{\pi}{4}-0\right)-\left(\frac{\pi}{8}-\frac{1}{4}\right)$

$=\frac{\pi}{8}+\frac{1}{4}$

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Q5 $x=-8$ $\qquad x^3-4y^2=12xy$

$(-8)^3-4y^2=12(-8)y$

$-512-4y^2=-96y \qquad /\div4$

$-128-y^2=-24y$

$y^2-24y+128=0$

$(y-16)(y-8)=0$

$y=8$ or $y=16$

(b) differentiating implicitly with respect to $x$

$3x^2-8y\frac{dy}{dx}=12y+12x\frac{dy}{dx}$

$12x\frac{dy}{dx}+8y\frac{dy}{dx}=3x^2-12y$

$\frac{dy}{dx}=\frac{3x^2-12y}{12x+8y}$

$\left.\frac{dy}{dx}\right|_{x=-8,y=8}=\frac{3(-8)^2-12(8)}{12(-8)+8(8)}=-3$ at $(-8,8)$ (gradient $=-3$)

$\left.\right|_{x=-8,y=16}=\frac{3(-8)^2-12(16)}{12(-8)+8(16)}=0$ at $(-8,16)$ (gradient $=0$)

Q6 $\underline{a} = 2i+6j-k \qquad \underline{b} = 3i+4j+k$

(a) $\vec{AB} = \underline{b}-\underline{a} = (3i+4j+k)-(2i+6j-k)$
$= i-2j+2k$

(b) $l_1 = 2i+6j-k + \lambda(i-2j+2k)$

or $l_1 = \begin{pmatrix}2\\6\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}$

(c) $\underline{l_2} = \gamma(i+k)$ or $l_2 = \gamma\begin{pmatrix}1\\0\\1\end{pmatrix}$

direction of $l_1$ : $d_1 = \begin{pmatrix}1\\-2\\2\end{pmatrix}$ of $l_2$: $d_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}$

dot product $\underline{d_1}\cdot\underline{d_2} = |d_1|\times|d_2|\cos\theta$

$\cos\theta = \dfrac{\begin{pmatrix}1\\-2\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{\left|\begin{pmatrix}1\\-2\\2\end{pmatrix}\right|\times\left|\begin{pmatrix}1\\0\\1\end{pmatrix}\right|} = \dfrac{(1)(1)+(-2)(0)+(2)(1)}{\sqrt{1^2+2^2+2^2}\times\sqrt{1^2+1^2}}$

$= \dfrac{1+0+2}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\theta = 45^\circ$

(c) $l_1 = l_2$

$\begin{pmatrix}2+\lambda\\6-2\lambda\\-1+2\lambda\end{pmatrix} = \begin{pmatrix}\gamma\\0\\\gamma\end{pmatrix} \Rightarrow$ $2+\lambda = \gamma$ (1)
$6-2\lambda = 0$ (2) $\Rightarrow \lambda = 3$ from (2)
$-1+2\lambda = \gamma$ (3) and $\gamma = 5$ from (1)
check with (3) $-1+6=5$ ✓

lies on $l_2$

Hence $\underline{c} = \gamma\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}5\\0\\5\end{pmatrix}$

Q7 $x = \ln(t+2)$, $y = \dfrac{1}{(t+1)}$ $\quad t > -1$

(a) Area $= \displaystyle\int_{\ln 2}^{\ln 4} y\,dx = \int_0^2 y\frac{dx}{dt}\,dt$ with the limits changed:

$x = \ln 2 \Rightarrow \ln 2 = \ln(t+2)$, $\underline{t=0}$

$x = \ln 4 \Rightarrow \ln 4 = \ln(t+2)$, $\underline{t=2}$

$\dfrac{dx}{dt} = \dfrac{1}{t+2}$

$= \displaystyle\int_0^2 \frac{dt}{(t+1)(t+2)}$

(4)

(b) $\dfrac{1}{(t+1)(t+2)} = \dfrac{A}{(t+1)} + \dfrac{B}{(t+2)}$

$= \dfrac{A(t+2)+B(t+1)}{(t+1)(t+2)}$

Comparing the numerators: $1 = At + \underline{2A} + Bt + \underline{B}$

$1 = 2A+B$
$0 = A+B$
$1 = A$
$B = -1$

$\displaystyle\int_0^2 \frac{dt}{(t+1)(t+2)} = \int\frac{dt}{t+1} + \int\frac{dt}{t+2}$

$= \Big[\ln(t+1) + \ln(t+2)\Big]_0^2$

$= \ln 3 + \ln 4 - (\ln 1 + \ln 2)$
$= \ln 3 + \ln 4 - \ln 2$

Q8 $\dfrac{dV}{dt} = 1600 - K\sqrt{h}$

constant rate at which it's being filled

leaking out, hence the MINUS and proportional to $\sqrt{h}$

(a) $V = 4000h$

area of cross section × height

$\dfrac{dh}{dt} = \dfrac{dh}{dV}\times\dfrac{dV}{dt}$ we already have $\dfrac{dV}{dt}$ so we need:

$\dfrac{dh}{dV} = \dfrac{1}{\frac{dV}{dh}} = \dfrac{1}{4000}$ $\qquad \dfrac{dV}{dh} = 4000$

$= \dfrac{1}{4000}\times(1600 - K\sqrt{h})$

$= 0.4 - \dfrac{K}{4000}\sqrt{h}$

Let $k = \dfrac{K}{4000}$ then $\underline{\dfrac{dh}{dt} = 0.4 - k\sqrt{h}}$ (3)

(b) leaking out rate is $K\sqrt{h}$
at $h = 25$ $\quad K\sqrt{25} = 400$
$K\times 5 = 400$
$K = 80$

Because $k = \dfrac{K}{4000} = \dfrac{80}{4000} = \underline{0.02}$

Q8 (c)

$\dfrac{dh}{dt} = 0.4 - 0.02\sqrt{h}$

$\displaystyle\int_0^{100}\frac{dh}{0.4-0.02\sqrt{h}} = \int dt$

$0.4 - 0.02\sqrt{h}$
$= 0.02\left(\frac{0.4}{0.02} - \sqrt{h}\right)$
$= 0.02(20-\sqrt{h})$

$\displaystyle\frac{1}{0.02}\int_0^{100}\frac{dh}{(20-\sqrt{h})} = \int dt$

$\dfrac{1}{0.02} = 50$

$\displaystyle 50\int_0^{100}\frac{dh}{(20-\sqrt{h})} = t.$

(d) $\displaystyle\int_0^{100}\frac{50\,dh}{20-\sqrt{h}}$

Let $u = 20-\sqrt{h}$ $\quad\{\sqrt{h} = 20-u$
$= 20 - h^{\frac{1}{2}}$

$\dfrac{du}{dh} = -\dfrac{1}{2}h^{-\frac{1}{2}} = -\dfrac{1}{2}\dfrac{1}{\sqrt{h}}$

Substitute for $20-\sqrt{h}$ and $dh$

$\{ dh = -2\sqrt{h}\,du$
$= -2(20-u)\,du$
$= 2(u-20)\,du$

$= \displaystyle\int_{u(0)}^{u(100)}\frac{100(u-20)}{u}\,du$

Change the limits $\quad h=0 \quad u=20$
$h=100 \quad u=10$

$= \displaystyle\int_{20}^{10}(100 - 2000u^{-1})\,du$

$\dfrac{50(u-20)}{u} = \dfrac{50u-1000}{u}$
$= 50 - 1000$

$= \Big[100u - 2000\ln u\Big]_{20}^{10}$

$= \{(100\times10 - 2000\ln 10\} - \{(100\times 20) - 2000\ln 20\}$

$1000 - 2000 - 2000\ln 10 + 2000(\ln 2 + \ln 10)$
$= -1000 + 2000\ln 2$